STABILITY OF AN EULERIAN ROD. NONLINEAR ANALYSIS

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1. Formulation of the Problem. The nonlinear Euler equation describing the deformed state of an elastic rod of variable stiffness, hinged at one end and subjected to a force P applied at the other end (Fig. 1), has the form [1, 2]

$$\frac{d^2y}{dx^2}+k^2\rho(x)\,y\left[1-\left(\frac{dy}{dx}\right)^2\right]^{0,5}\equiv F(y,k)=0,$$
$$x_0\leqslant x\leqslant x_1=x_0+\pi,\quad k^2=\frac{Pl^2}{EI\pi^2}, \tag{1.1}$$

where y is the deflection, E is Young's modulus, I is the moment of inertia, the function $\rho(x)$ characterizes the variation of the stiffness along the rod, ℓ is the length of the rod, $F(y, k) = 0$ is the operator form of Euler's equation, and the rest of the notation is elucidated in Fig. 1.

Equation (1.1) is an example that is quite often encountered in the proof of existence theorems for solutions of nonlinear differential equations [1-3] and qualitative analysis of many physical phenomena, modeled by these equations [4, 5]

In applied analysis the linearized Euler equation is employed for solving different particular problems, differing by the method employed for fastening the rod and the character of the forces acting on the rod [6-8]. The nonlinear equation (1.1) has been solved in elliptic functions only for a rod with a constant transverse cross section ($\rho(x) = 1$) [3].

In some works [6, 8] the displacement Δ of the free end of the rod is neglected as a higher-order infinitesimal compared with the deflection, and Eq. (1.1) is replaced by the nonlinear equation

$$\frac{d^2y}{ds^2}+k^2\rho(s)\,y\left[1+\left(\frac{dy}{ds}\right)^2\right]^{1,5}=0, \tag{1.2}$$

which is solved under the assumption that $\rho(s) = 1$.

Equations (1.1) and (1.2) become identical after linearization, but large errors are made in nonlinear analysis if Eq. (1.2) is used instead of Eq. (1.1). However, the assertion made in [2] that Eq. (1.2) has no solutions for $k^2 > 0$ and, therefore, that this equation is physically meaningless is not absolute. This will be shown below.

The stability of the solutions of Eq. (1.1) with the boundary conditions

$$y(x_0)=y(x_1)=0 \tag{1.3}$$

is investigated by the method of projections [4], according to which the complete space of characteristic functions of an appropriate linear operator is determined and the concept of amplitude is introduced. The condition of solvability (Fredholm's theorem of the alternative) enables calculation of the boundary dividing the range of the parameters of the problem (1.1) and (1.3) into zones of stable and unstable solutions.

Besides Eq. (1.1), the stability of solutions of the equation for a loaded rod containing imperfections has also been studied.

Completing the formulation of the problem, we note that if the boundary conditions for Eq. (1.2) are

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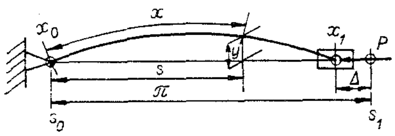

Fig. 1

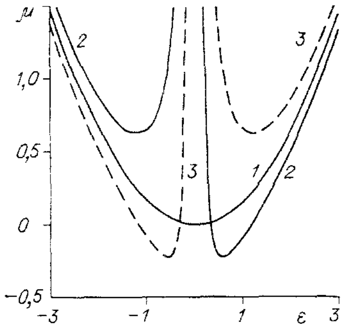

Fig.2

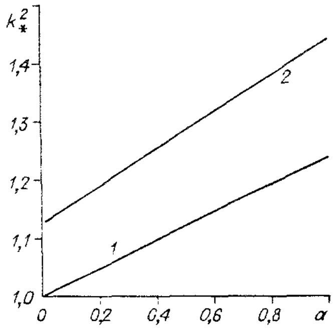

Fig. 3

$$y(s_0) = y(s_1 - \Delta(y)) = 0, \tag{1.4}$$

then the problems (1.1), (1.3) and (1.2), (1.4) are equivalent. In order to analyze stability it is preferable to write the problem in the form (1.1) and (1.3) due to the homogeneity of the boundary conditions.

2. Stability of Analysis. Expanding Eq. (1.1) in a series in powers of y and dy/dx at the point (y, dy/dx) = (0, 0) gives

$$F(y, k) \equiv L_k y + \sum_{n=2}^{\infty} c_n y (dy/dx)^{n-1} = 0, \tag{2.1}$$

where $L_k = d^2/dx^2 + k^2\rho(x)$ is the generating operator;

$$c_n = \frac{1}{n!} \frac{\partial^n}{\partial y\, \partial (dy/dx)^{n-1}} \left(k^2 \rho(x)\, y\, (1 - (dy/dx)^2)^{0,5}\right).$$

The spectrum of the operator L_k consists of the eigenvalues λ^2 of the boundary-value problem

$$L_\lambda y = 0, \quad y(x_0) = y(x_1) = 0. \tag{2.2}$$

In the general case, if $\rho(x)$ does not have some special form, the eigenvalue problem (2.2) can be solved only by approximate methods [9]. The question of convergence of any method remains open and depends on making a successful choice of the basic equation. Without making any claims as to the generality or completeness of the analysis, but rather starting from the fact that k^2 can always be those (the moment of inertia of the maximum transverse section of the rod can be chosen) so that $0 < \rho(x) \leq 1$ on the interval $x_0 \leq x \leq x_1$, we take as the basic equation

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0, \quad y(x_0) = y(x_1) = 0. \tag{2.3}$$

Then, using the fundamental system of solutions of the problem (2.3), consisting of the functions $\sin\lambda x$ and $\cos\lambda x$, and bearing in mind the first boundary condition, it is convenient to write Eq. (2.2) as an equivalent integral equation

$$\varphi(x) = \sin\lambda(x - x_0) - \lambda A\varphi(x),$$
$$A\varphi(x) = \int_{x_0}^{x} \sin\lambda(x - \xi)(\rho(\xi) - 1)\varphi(\xi)\, d\xi. \tag{2.4}$$

The eigenvalues λ_1^2, λ_2^2... and the corresponding eigenfunctions $\varphi_1(x)$, $\varphi_2(x)$, ... of Eq. (2.4) will also be eigenvalues and eigenfunctions of Eq. (2.2). It is important to note that zero is not an eigenvalue, and that the nonzero eigenvalues are simple. (For some special functions $\rho(x)$ the problem (2.2) does have an exact solution, and the eigenvalues of the operator L_λ are doubly degenerate. This case is not considered here.)

Equation (2.4) is solved by the method of successive approximations

$$\varphi^{(n)}(x) = \sin\lambda(x - x_0) - \lambda A\varphi^{(n-1)}(x), \tag{2.5}$$

and at each step the eigenvalue λ^2 is determined from the equation obtained taking into account the second boundary condition,

$$\sin\lambda\pi - A\varphi^{(n)}(x_1) = 0. \tag{2.6}$$

As the zeroth approximation we can take $\varphi^{(0)}(x) = 0,\ 1$ or any other function satisfying the boundary conditions (1.3). The principle of uniform boundedness [1, 2] guarantees that the sequence $\varphi^{(n)}(x)$ converges, and since $0 < \rho(x) \leqslant 1$, it is obvious that the rate of convergence will be characterized by the difference $|\rho_1 - 1|$, where ρ_1 is the minimum value of the function $\rho(x)$ on the interval (x_0, x_1). Indeed, since

$$\sup_{\substack{x_0 \leqslant x \leqslant x_1 \\ x_0 \leqslant \xi \leqslant x_1}} |\sin\lambda(x - \xi)(\rho(\xi) - 1)| = |\rho_1 - 1|,$$

$$\sup_{x_0 \leqslant x \leqslant x_1} |\sin\lambda(x - x_0)| = 1,$$

we have the estimate [10]

$$|\varphi^{(n+1)}(x) - \varphi^{(n)}(x)| \leqslant |\lambda|^n |\rho_1 - 1|^n \frac{(x - x_0)^n}{n!}.$$

Finally, if the integral equation (2.4) does not have an exact solution, which is most likely the case, then it is convenient to represent $\rho(x)$ as an expansion in eigenfunctions of Eq. (2.3):

$$\rho(x) = \sum_{k=1}^{\infty} b_k \sin kx, \quad b_k = \frac{2}{\pi}\int_{x_0}^{x_1} \rho(x)\sin kx\, dx.$$

Having determined the eigenfunctions and eigenvalues of the operator L_k, we note that the minimum positive eigenvalue λ_1^2 enables writing the stability condition for the solution (2.2) as

$$\mu = k^2 - \lambda_1^2 \leqslant 0. \tag{2.7}$$

The equal sign in Eq. (2.7) determines the critical value k_*^2, obtained in the linear approximation. For $\rho(x) = 1$ the result $\mu = k_*^2 - 1 = 0$ is well known [6-8].

Returning to the functions $\varphi_i(x),\ i = 1, 2, \ldots$, it is easy to show that they are square-integrable on the interval (x_0, x_1) and are orthogonal with respect to one another with $\rho(x)$ as the weighting function. The space of these functions is complete with the scalar product $\langle\varphi_i(x), \varphi_j^*(x)\rangle$ of the vectors $\varphi_i(x)$ and $\varphi_j(x)$ (all properties of a Hilbert space), so that any solutions of Eq. (2.1) can be represented as expansions in $\varphi_1(x), \varphi_2(x), \ldots$

The function $\varphi_j^*(x)$, which has appeared above, is the conjugate of the function $\varphi_j(x)$ with respect to the scalar product and is equal to $\varphi_j^*(x) = M\rho(x)\varphi_j(x)$, where M is a constant factor.

Now, having determined the amplitude as the projection of y(x) on the characteristic subspace associated with the conjugate vector $\varphi_1^*(x)$, $\varepsilon = \langle y(x), \varphi_1^*(x)\rangle$, we seek the solution of Eq. (2.1) in the form of the series

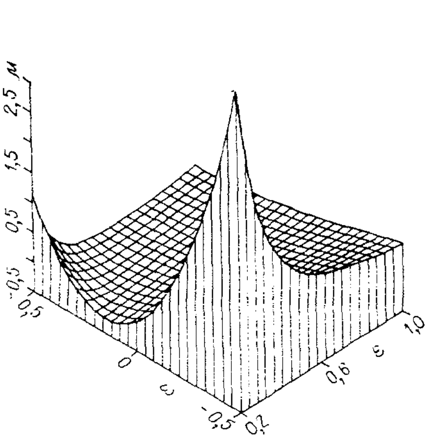

Fig. 4

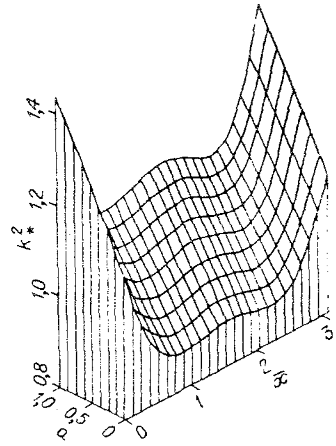

Fig. 5

$$\begin{vmatrix} y \\ \mu \end{vmatrix} = \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \begin{vmatrix} y_n \\ \mu_n \end{vmatrix}. \tag{2.8}$$

Before determining y_n, μ_n, $n = 1, 2, \ldots$, we express in Eq. (2.1), using Eq. (2.7) with the equal sign, k^2 in terms of μ and λ_1^2 so that the operator is

$$L_\mu = d^2/dx^2 + (\mu + \lambda_1^2)\rho(x).$$

Then substituting Eq. (2.8) into Eq. (2.1) and equating terms with like powers of ε up to the cubic terms inclusively gives the system

$$L_0 y_1 = 0; \tag{2.9}$$

$$L_0 y_2 + 2\mu_1 \frac{\partial L_0}{\partial \mu} y_1 + 2\mathbf{B}(\mathbf{y}_1, \mathbf{y}_1) = 0; \tag{2.10}$$

$$L_0 y_3 + 3\mu_1 \frac{\partial L_0}{\partial \mu} y_2 + 6\mathbf{B}(\mathbf{y}_1, \mathbf{y}_2) + 3\mu_2 \frac{\partial L_0}{\partial \mu} y_1 + 6\mathbf{C}(\mathbf{y}_1, \mathbf{y}_1, \mathbf{y}_1) = 0, \tag{2.11}$$

where

$$\mathbf{B}(\mathbf{y}_1, \mathbf{y}_2) = \frac{c_2 \lambda_1^2}{2}\left(y_1 \frac{dy_2}{dx} + y_2 \frac{dy_1}{dx}\right);$$

$$\mathbf{C}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) = \frac{c_3 \lambda_1^2}{3}\left(\mathbf{y}_1 \frac{dy_2}{dx}\frac{dy_3}{dx} + y_2 \frac{dy_1}{dx}\frac{dy_3}{dx} + y_3 \frac{dy_1}{dx}\frac{dy_2}{dx}\right).$$

It follows directly from Eq. (2.9) that $y_1 = \varphi_1(x)$. Equations (2.10) and (2.11) are solved with the help of Fredholm's theorem of the alternative, according to which these equations are solvable if

$$\langle L_0 y_2, \varphi_1^*(x)\rangle = \langle L_0 y_3, \varphi_1^*(x)\rangle = 0$$

and hence

$$\mu_1 \left\langle \frac{\partial L_0}{\partial \mu} y_1, \varphi_1^*(x)\right\rangle + \langle \mathbf{B}(\mathbf{y}_1, \mathbf{y}_1), \varphi_1^*(x)\rangle = 0,$$

$$\mu_1 \left\langle \frac{\partial L_0}{\partial \mu} y_2, \varphi_1^*(x)\right\rangle + 2\langle \mathbf{B}(\mathbf{y}_1, \mathbf{y}_2), \varphi_1^*(x)\rangle +$$
$$+ \mu_2 \left\langle \frac{\partial L_0}{\partial \mu} y_1, \varphi_1^*(x)\right\rangle + 2\langle \mathbf{C}(\mathbf{y}_1, \mathbf{y}_1, \mathbf{y}_1), \varphi_1^*(x)\rangle = 0,$$

whence, since $c_2 = 0$ and $c_3 = -0.5$, it is easily found that $\mu_1 = 0$, $y_2 = 0$, and

$$\mu_2 = \frac{\lambda_1^2 \langle \rho(x) y_1 (dy_1/dx)^2, \varphi_1^*(x)\rangle}{\langle \rho(x) y_1, \varphi_1^*(x)\rangle}.$$

Thus the stability boundary in the (μ, ε) plane is given by the equation

$$\mu = 0{,}5\mu_2\varepsilon^2. \tag{2.12}$$

In order that the solution (2.12) be unique, it is necessary to find a normalization condition. To this end, we substitute Eq. (2.8) into the expression for the amplitude $\varepsilon = \langle y, \varphi_1^*(x)\rangle$ and differentiate the latter with respect to ε. The result is the equation

$$\left\langle \sum_{k=1}^{\infty} \frac{1}{(k-1)!} y_k \varepsilon^{k-1}, \varphi_1^*(x) \right\rangle = 1,$$

which, since $\langle y_k, \varphi_1^*(x)\rangle = 0$ for $k \neq 1$, gives the normalization condition $\langle y_1, \varphi_1^*(x)\rangle = 1$. This condition is equivalent to $\varepsilon = 1$, $M = \langle \varphi_1(x), \rho(x)\varphi_1(x)\rangle^{-1}$ and makes it possible to determine the critical value k^2_* for which the solution of Eq. (2.1) becomes unstable:

$$k_*^2 = \lambda_1^2 + 0{,}5\mu_2. \tag{2.13}$$

Since $\mu_2 > 0$ for any function $\rho(x)$, the critical load $P_* = P(k_*^2)$ in nonlinear analysis is higher than that obtained in the linear approximation.

Returning to Eq. (1.2), we note that this equation has no solutions for $k^2 > 0$, if $\lambda_1^2 < 1.5\mu_2$.

As an example, consider a rod with a parabolic transverse cross section

$$\rho(x) = \pi^2(\pi^2 + ax(\pi - x))^{-1}, \quad 0 \leqslant x \leqslant \pi,$$

where a is a constant. As expected, the integral equation (2.4) does not have an exact solution. For this reason we represent $\rho(x)$ by the first two terms of its series expansion:

$$\rho(x) = b_1 \sin x + b_2 \sin 2x.$$

Substituting this expression into Eq. (2.5) and writing $\varphi^{(0)}(x) = 1$, we find the first approximation for the eigenfunction:

$$\varphi_1(x) = \varphi_1^{(1)}(x) = \sin\lambda_1 x + \lambda_1 \left[\frac{b_1}{1-\lambda_1^2}(\lambda_1 \sin x - \sin\lambda_1 x) + \right.$$
$$\left. + \frac{2b_2}{4-\lambda_1^2}(\lambda_1 \sin 2x - \sin\lambda_1 x) \right]. \tag{2.14}$$

Substituting Eq. (2.14) into Eq. (2.6) gives an equation for λ_1:

$$1 + \lambda_1 \left[\frac{b_1}{\lambda_1^2 - 1} + \frac{2b_2}{\lambda_1^2 - 4} \right] = 0. \tag{2.15}$$

The results of calculations of the stability boundary in the (μ, ε) plane, performed using the formula (2.12) and Eqs. (2.14) and (2.15) with $a = 0.5$, are displayed in Fig. 2 (curve 1); Fig. 3 displays the function $k_*^2 = k_*^2(a)$, calculated using Eqs. (2.7) and (2.13) (curves 1 and 2, respectively). For $a = 0$, calculations using Eq. (2.7) give $k_*^2 = 1$, and Eq. (2.13) gives $k_*^2 = 1.125$. For comparison, we indicate that for Eq. (1.2) the formula (2.13) with $a = 0$ gives $k_*^2 = 0.625$.

3. Imperfections. Let the rod contain imperfections such that the problem is written mathematically in the form

$$\frac{d^2y}{dx^2} + k^2 \left[\rho(x) y (1 - (dy/dx)^2)^{0,5} + \omega \sum_{n=0}^{\infty} \psi_n(x) y^n \right] \equiv F(y, \mu, \omega) = 0,$$
$$k^2 = \mu + \lambda_1^2, \; x_0 \leqslant x \leqslant x_1 = x_0 + \pi, \quad y(x_0) = y(x_1) = 0. \tag{3.1}$$

where ω is a constant; $\psi_n(x)$ $(n = 0, 2, \ldots)$ are functions, among which at least $\psi_0(x)$ is not identically zero.

The meaning of Eq. (3.1) is that in the absence of a load $k^2 \to 0$, $\omega \sim 1/k^2$ the axis of the rod is not an ideal straight line, and is determined by the equation

$$\frac{d^2y}{dx^2} + k^2\omega \sum_{n=0}^{\infty} \psi_n(x)\, y^n = 0.$$

As one can see from Sec. 2, the solution of the equation $F(y, \mu, 0) = 0$ becomes unstable when μ changes from negative to positive. Hence [4, 11], the point $(y, \mu) = (0, 0)$ is a double singular point, where branching of the solutions occurs. The presence of imperfections $\omega \neq 0$, which destroy the bifurcation at the point $(y, \mu) = (0, 0)$, leads to isolated solutions of the equation $F(y, \mu, \omega) = 0$ to which the point $(y, \mu) = (0, 0)$ does not belong.

The condition $\langle \partial F(0, 0, 0)/\partial\omega,\ \varphi_1^*(x)\rangle \neq 0$ (we use below the simplified notation $F(0, 0, 0) = F$) and the implicit function theorem guarantee that a solution of the equation

$$F(y(\mu, \varepsilon),\ \mu,\ z(\mu, \varepsilon)) = 0 \tag{3.2}$$

for the formally introduced function $z(\mu, \varepsilon) = \omega$ exists.

This solution will be sought as a series in powers of μ, ε at the point $(\mu, \varepsilon) = (0, 0)$. Once again, simplifying the notation by writing $z = z(0, 0)$, we set

$$\begin{aligned} z(\mu, \varepsilon) = z + \frac{\partial z}{\partial\varepsilon}\varepsilon + \frac{\partial z}{\partial\mu}\mu + \frac{1}{2}\left(\frac{\partial^2 z}{\partial\varepsilon^2}\varepsilon^2 + 2\frac{\partial^2 z}{\partial\mu\,\partial\varepsilon}\mu\varepsilon + \frac{\partial^2 z}{\partial\mu^2}\mu^2\right) + \\ + \frac{1}{3!}\left(\frac{\partial^3 z}{\partial\varepsilon^3}\varepsilon^3 + 3\frac{\partial^3 z}{\partial\mu\,\partial\varepsilon^2}\mu\varepsilon^2 + 3\frac{\partial^3 z}{\partial\mu^2\partial\varepsilon}\mu^2\varepsilon + \frac{\partial^3 z}{\partial\mu^3}\mu^3\right) + \ldots \end{aligned} \tag{3.3}$$

In order to determine the coefficients in the expansion (3.3), as a start, we employ the properties of the double point

$$F = 0, \quad \frac{\partial F}{\partial\varepsilon} - \frac{\partial \mathbf{F}}{\partial y}\left(\frac{\partial \mathbf{y}}{\partial\varepsilon}\right) - \frac{\partial F}{\partial z}\frac{\partial z}{\partial\varepsilon} = 0,$$
$$\frac{\partial F}{\partial\mu} + \frac{\partial \mathbf{F}}{\partial y}\left(\frac{\partial \mathbf{y}}{\partial\mu}\right) + \frac{\partial F}{\partial z}\frac{\partial z}{\partial\mu} = 0.$$

The first relation gives, using Eq. (2.8), $z = 0$, and from the conditions that the remaining two relations have a solution

$$\left\langle \frac{\partial F}{\partial z}\frac{\partial z}{\partial\varepsilon},\ \varphi_1^*(x)\right\rangle = \left\langle \frac{\partial F}{\partial z}\frac{\partial z}{\partial\mu},\ \varphi_1^*(x)\right\rangle = 0$$

and the inequality $\partial F/\partial z \neq 0$ it follows that $\partial z/\partial\mu = \partial z/\partial\varepsilon = 0$.

The second derivatives of Eq. (3.2) lead to the system

$$\frac{\partial \mathbf{F}}{\partial y}\left(\frac{\partial^2 \mathbf{y}}{\partial\varepsilon^2}\right) + \frac{\partial^2 \mathbf{F}}{\partial y^2}\left(\frac{\partial \mathbf{y}}{\partial\varepsilon}, \frac{\partial \mathbf{y}}{\partial\varepsilon}\right) + \frac{\partial F}{\partial\omega}\frac{\partial^2 z}{\partial\varepsilon^2} = 0,$$
$$\frac{\partial \mathbf{F}}{\partial y}\left(\frac{\partial^2 \mathbf{y}}{\partial\mu\,\partial\varepsilon}\right) + \frac{\partial^2 \mathbf{F}}{\partial\mu\,\partial y}\left(\frac{\partial \mathbf{y}}{\partial\varepsilon}\right) + \frac{\partial F}{\partial\omega}\frac{\partial^2 z}{\partial\mu\,\partial\varepsilon} = 0,$$
$$\frac{\partial \mathbf{F}}{\partial y}\left(\frac{\partial^2 \mathbf{y}}{\partial\mu^2}\right) + \frac{\partial^2 F}{\partial\mu^2} + \frac{\partial F}{\partial\omega}\frac{\partial^2 z}{\partial\mu^2} = 0,$$

the solvability condition for which

$$\left\langle \frac{\partial \mathbf{F}}{\partial y}\left(\frac{\partial^2 \mathbf{y}}{\partial \varepsilon^2}\right), \quad \varphi_1^*(x) \right\rangle = \left\langle \frac{\partial \mathbf{F}}{\partial y}\left(\frac{\partial^2 \mathbf{y}}{\partial \mu\, \partial \varepsilon}\right), \quad \varphi_1^*(x) \right\rangle =$$
$$= \left\langle \frac{\partial \mathbf{F}}{\partial y}\left(\frac{\partial^2 \mathbf{y}}{\partial \mu^2}\right), \quad \varphi_1^*(x) \right\rangle = 0$$

give, using the effect that $c_2 = 0,\ \partial^2 F/\partial \mu^2 = 0$,

$$\frac{\partial^2 z}{\partial \varepsilon^2} = 0, \quad \frac{\partial^2 y}{\partial \varepsilon^2} = 0, \quad \frac{\partial^2 y}{\partial \mu^2} = 0, \quad \frac{\partial^2 z}{\partial \mu^2} = 0,$$
$$\frac{\partial^2 z}{\partial \mu\, \partial \varepsilon} = -\left\langle \frac{\partial^2 \mathbf{F}}{\partial y\, \partial \mu}\left(\frac{\partial \mathbf{y}}{\partial \varepsilon}\right), \ \varphi_1^*(x) \right\rangle \left\langle \frac{\partial F}{\partial \omega}, \ \varphi_1^*(x) \right\rangle^{-1}. \tag{3.4}$$

Finally, the third derivatives of Eq. (3.2), in which terms containing zero cofactors are dropped, show that the system of equations

$$\frac{\partial \mathbf{F}}{\partial y}\left(\frac{\partial^3 \mathbf{y}}{\partial \varepsilon^3}\right) + \frac{\partial F}{\partial \omega}\frac{\partial^3 z}{\partial \varepsilon^3} + \frac{\partial^3 \mathbf{F}}{\partial y^3}\left(\frac{\partial \mathbf{y}}{\partial \varepsilon}, \frac{\partial \mathbf{y}}{\partial \varepsilon}, \frac{\partial \mathbf{y}}{\partial \varepsilon}\right) = 0,$$
$$\frac{\partial \mathbf{F}}{\partial y}\left(\frac{\partial^3 \mathbf{y}}{\partial \mu\, \partial \varepsilon^2}\right) + \frac{\partial F}{\partial \omega}\frac{\partial^3 z}{\partial \mu\, \partial \varepsilon^2} + 2\frac{\partial^2 \mathbf{F}}{\partial \omega\, \partial y}\left(\frac{\partial \mathbf{y}}{\partial \varepsilon}\right)\frac{\partial^2 z}{\partial \mu\, \partial \varepsilon} = 0,$$
$$\frac{\partial \mathbf{F}}{\partial y}\left(\frac{\partial^3 \mathbf{y}}{\partial \mu^2 \partial \varepsilon}\right) + \frac{\partial F}{\partial \omega}\frac{\partial^3 z}{\partial \mu^2 \partial \varepsilon} + 2\frac{\partial^2 \mathbf{F}}{\partial \mu\, \partial y}\left(\frac{\partial^2 \mathbf{y}}{\partial \mu\, \partial \varepsilon}\right) + 2\frac{\partial^2 F}{\partial \mu\, \partial \omega}\frac{\partial^2 z}{\partial \mu\, \partial \varepsilon} = 0,$$
$$\frac{\partial \mathbf{F}}{\partial y}\left(\frac{\partial^3 \mathbf{y}}{\partial \mu^3}\right) + \frac{\partial F}{\partial \omega}\frac{\partial^3 z}{\partial \mu^3} = 0$$

has a solution if

$$\frac{\partial^3 z}{\partial \varepsilon^3} = -\left\langle \frac{\partial^3 \mathbf{F}}{\partial y^3}\left(\frac{\partial \mathbf{y}}{\partial \varepsilon}, \frac{\partial \mathbf{y}}{\partial \varepsilon}, \frac{\partial \mathbf{y}}{\partial \varepsilon}\right), \ \varphi_1^*(x) \right\rangle \left\langle \frac{\partial F}{\partial \omega}, \ \varphi_1^*(x) \right\rangle^{-1},$$
$$\frac{\partial^3 z}{\partial \mu\, \partial \varepsilon^2} = -2\left\langle \frac{\partial^2 \mathbf{F}}{\partial \omega\, \partial y}\left(\frac{\partial \mathbf{y}}{\partial \varepsilon}\right)\frac{\partial^2 z}{\partial \mu\, \partial \varepsilon}, \ \varphi_1^*(x) \right\rangle \left\langle \frac{\partial F}{\partial \omega}, \ \varphi_1^*(x) \right\rangle^{-1},$$
$$\frac{\partial^3 z}{\partial \mu^2 \partial \varepsilon} = -2\left\langle \frac{\partial^2 F}{\partial \mu\, \partial \omega}\frac{\partial^2 z}{\partial \mu\, \partial \varepsilon}, \varphi_1^*(x) \right\rangle \left\langle \frac{\partial F}{\partial \omega}, \varphi_1^*(x) \right\rangle^{-1}, \quad \frac{\partial^3 z}{\partial \mu^3} = 0. \tag{3.5}$$

The derivatives $\partial^n F/\partial y^i \ (i \leqslant n)$ in the expressions presented above must be interpreted as matrix differential operators (Fréchet derivatives), so that we write the derivatives appearing in the system of equations (3.5) as

$$\frac{\partial^3 \mathbf{F}}{\partial y^3}\left(\frac{\partial \mathbf{y}}{\partial \varepsilon}, \frac{\partial \mathbf{y}}{\partial \varepsilon}, \frac{\partial \mathbf{y}}{\partial \varepsilon}\right) = \mathbf{C}(\mathbf{y}_1, \mathbf{y}_1, \mathbf{y}_1),$$
$$\frac{\partial^2 \mathbf{F}}{\partial \omega\, \partial y}\left(\frac{\partial \mathbf{y}}{\partial \varepsilon}\right) = \lambda_1^2 \psi_1(x)\, y_1, \ \frac{\partial^2 \mathbf{F}}{\partial y\, \partial \mu}\left(\frac{\partial \mathbf{y}}{\partial \varepsilon}\right) = \rho(x)\, y_1.$$

After substituting the expressions for the coefficients (3.4) and (3.5) into Eq. (3.3), the latter equation assumes the form

$$\omega \langle \psi_0(x), \varphi_1^*(x) \rangle = -\lambda_1^{-2} \langle \rho(x)\, y_1, \varphi_1^*(x) \rangle \mu\varepsilon +$$
$$+ 0{,}5 \langle \rho(x)\, y_1 (dy_1/dx)^2, \varphi_1^*(x) \rangle \varepsilon^3 - \lambda_1^2 \langle \psi_1(x)\, y_1, \varphi_1^*(x) \rangle \langle \rho(x)\, y_1, \varphi_1^*(x) \rangle \times$$
$$\times \langle \psi_0(x), \varphi_1^*(x) \rangle^{-1} \mu\varepsilon^2 + \lambda_1^{-4} \langle \rho(x)\, y_1, \varphi_1^*(x) \rangle \mu^2 \varepsilon. \tag{3.6}$$

Solving Eq. (3.6) by the method of successive approximations, we find the following relations between the parameters $\mu = \mu(\varepsilon, \omega/\varepsilon)$:

$$\mu=\lambda_1^2\left[\frac{\langle\rho(x)y_1(dy_1/dx)^2,\ \varphi_1^*(x)\rangle}{2\langle\rho(x)y_1,\ \varphi_1^*(x)\rangle}\varepsilon^2-\frac{\langle\psi_0(x),\ \varphi_1^*(x)\rangle}{\langle\rho(x)y_1,\ \varphi_1^*(x)\rangle}\frac{\omega}{\varepsilon}-\right.$$

$$\left.-\frac{\langle\psi_1(x)y_1,\ \varphi^*(x)\rangle}{\langle\rho(x)y_1,\ \varphi_1^*(x)\rangle}\frac{\omega}{\varepsilon}\varepsilon+\frac{\langle\psi_0(x),\ \varphi_1^*(x)\rangle^2}{\langle\rho(x)y_1,\ \varphi_1^*(x)\rangle^2}\left(\frac{\omega}{\varepsilon}\right)^2\right]+O\left(|\varepsilon|+\left|\frac{\omega}{\varepsilon}\right|\right)^3. \quad (3.7)$$

In the absence of imperfections ($\omega = 0$) Eqs. (3.7) and (2.12) are identical. Substituting into Eq. (3.7) the expression for μ (2.7) and using the normalization condition $\varepsilon = 1$ we obtain the relation for the critical value k_*^2 of a rod with imperfections.

Now assume that the rod studied as an example in Sec. 2 contains at the point $x=\bar{x}$ a defect, not depending on the deflection, and let $\psi_0(x)=\delta(x-\bar{x})$ – a delta function and $\psi_i(x)=0,\ i>0$.

Substituting the expression for $\psi_0(x)$ into Eq. (3.7) and performing calculations with $\bar{x}=1.5,\ a=0.5$ gives the functions $\mu = \mu(\varepsilon,\ \omega/\varepsilon)$, displayed in Fig. 4, and $\mu=\mu(\varepsilon;\ 0.5/\varepsilon)$, $\mu=\mu(\varepsilon;\ -0.5/\varepsilon)$, displayed in Fig. 2 (curves 2 and 3, respectively).

Figure 5 displays the computational results obtained for the critical value $k_*^2=k_*^2(\bar{x},a)$ using Eq. (3.7) with $\omega = 1$. For $\bar{x}=0,\ \bar{x}=\pi$ the calculations using Eqs. (2.13) and (3.7) give the same results.

In conclusion it should be noted that nonlinear analysis is useful in studying the the theory of stability of an Eulerian rod as a subfield of strength calculations in machine building, especially in cases of structures consisting of thin-walled rods, for which the limiting loads are limited by considerations of stability.

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